Lecture 12. Reduction of Order and Euler Equations

The method of reduction of order

Suppose that one solution $y_1(x)$ of the homogeneous second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

is known (on an interval I where p and q are continuous functions).

The method of **reduction of order** consists of substituting $y_2(x) = v(x)y_1(x)$ in (1) and determine the function $v(x)$ so that $y(x)$ is a second linearly independent solution of (1).

After substituting $y_2(x) = v(x)y_1(x)$ in Eq. (1), use the fact that $y_1(x)$ is a solution. We can deduce that

$$y_1v'' + (2y_1' + py_1)v' = 0$$

We can solve this for v to find the solution $y_2(x)$ of equation (1).

Example 1 Consider the equation

$$x^2y'' - 5xy' + 9y = 0 \ (x > 0),$$

Notice that $y_1(x) = x^3$ is a solution. Subsitute $y = vx^3$ and deduce that $xv'' + v' = 0.$

Solve this equation and obtain the second solution $y_2(x) = x^3 \ln x.$

ANS: We write the given eqn in the form of Eq (1).

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0 \quad \circledast$$

Let $y_2 = v(x)y_1(x) = v(x)x^3$

$$y_2' = v'x^3 + 3vx^2$$

$$y_2'' = v''x^3 + 3v'x^2 + 3v'x^2 + 6vx$$

$$= v''x^3 + 6v'x^2 + 6xv$$

Plug them into $\circledast$.

$$y_2'' - \frac{5}{x}y_2' + \frac{9}{x^2}y_2 = 0$$

$$\Rightarrow v''x^3 + 6v'x^2 + 6xv - \frac{5}{x}(v'x^3 + 3vx^2) + \frac{9}{x^2}v \cdot x^3 = 0$$

$$\Rightarrow v''x^3 + 6v'x^2 + \cancel{6xv} - 5v'x^2 - \cancel{15vx} + \cancel{9vx} = 0$$

$$\Rightarrow v''x^3 + v'x^2 = 0$$

$$\Rightarrow x^2(v''x + v') = 0$$

$$\Rightarrow v''x + v' = 0 \quad ☆$$

Note there is no term in ☆ about $v(x)$.

So we can introduce

$$u(x) = v', \text{ then } v'' = u'$$

Plug them into ☆. we have

$$u'x + u = 0 \quad (\text{sep. eqn})$$

$$\Rightarrow \frac{du}{dx} x = -u$$

$$\Rightarrow \int \frac{du}{u} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln|u| = -\ln x + C_1$$

$$\Rightarrow e^{\ln|u|} = e^{-\ln x + c_1} = c_2 e^{-\ln x}$$

$$\Rightarrow u = C e^{-\ln x} = \frac{C}{x}$$

Recall $u = v'$, we have

$$v' = \frac{dv}{dx} = u = \frac{c}{x} \Rightarrow v(x) = \int \frac{c}{x} dx$$

$$\Rightarrow v(x) = C \ln x + C_3$$

Note it sufficies to find one $v(x)$, so we can take a simple form of $v(x)$ by assuming. $C=1$, $C_3 = 0$

Then $v(x) = \ln x$

So $y_2(x) = v(x) y_1 = x^3 \ln x$ is another solution to ⊛.

Remark: The method still works if the eqn is not in the form of Eq(1). See Exercise 2.

Exercise 2. The differential equation

$$x^2\frac{d^2y}{dx^2}-7x\frac{dy}{dx}+16y=0 \qquad \circledast$$

has $y_1(x)=x^4$ as a solution. Use the method of reduction of order to find a second solution $y_2(x)$.

ANS: Let $y_2 = v(x)y_1(x) = vx^4$

Then we calculate:

$$y_2' = (vx^4)' = v'x^4 + 4vx^3$$

$$y_2'' = (v'x^4+4vx^3)' = v''x^4 + 4v'x^3 + 4v'x^3 + 12vx^2$$

$$= v''x^4 + 8v'x^3 + 12vx^3$$

Plug y_2, y_2', y_2'' into $\circledast$, we have

$$x^2y_2'' - 7xy_2' + 16y_2$$

$$= x^2(v''x^4+8v'x^3+12vx^2) - 7x(v'x^4+4vx^3) + 16vx^4$$

$$= v''x^6 + 8v'x^5 + 12vx^4 - 7v'x^5 - 28vx^4 + 16vx^4$$

$$= v''x^6 + v'x^5$$

$$= 0$$

Thus we have

$$x^2y_2'' - 7xy_2' + 16y_2 = x^6v'' + x^5v' = 0$$

$$\Rightarrow \quad xv'' + v' = 0 \qquad \circledast\circledast$$

Notice that the above equation does not have any terms $v(x)$, we can make a substitution

$$w = v'$$

Then $v'' = w'$

Plug $v'' = w'$, $v' = w$ into ⊗⊗, we have

$$xw' + w = 0 \quad \text{or} \quad w' + \frac{1}{x} w = 0 \quad (x \neq 0)$$

which is a separable equation. (or you can solve it as a linear first order eqn)

The webwork HW asks us to solve it as linear 1st order eqn.

An integrating factor is $p(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ (assuming $x > 0$)

Thus we have $p(x) w = xw = \int 0 \, dx = a$ (where a is any constant)

$$\Rightarrow w = \frac{a}{x}$$

Recall

$$w = v' = \frac{a}{x}$$

Integrate both sides, we get

$$v = \int \frac{a}{x} dx = a \ln x + b$$

Thus $y_2 = v(x) y_1(x) = (a \ln x + b) x^4 = a x^4 \ln x + b x^4$ is a second solution, where a and b are constants.

The general solution is

$y = c_1 y_1 + c_2 y_2 = c_1 x^4 + c_2 (a x^4 \ln x + b x^4)$, which is basically a linear combination of x^4 and $x^4 \ln x$

So we can state the general solution is

$$y = a x^4 \ln x + b x^4$$

Euler Equation

A second-order Euler equation is one of the form

$$ax^2y'' + bxy' + cy = 0 \tag{2}$$

where a, b, c are constants.

Remark. Note the previous example and exercise are also Euler equations. So the method below also works for **Example 1** and **Exercise 2**.

Example 3. Make the substitution $v = \ln x$ of the following question to find general solutions (for $x > 0$) of the Euler equation.

$$x^2y'' + 2xy' - 12y = 0 \qquad (3)$$

ANS: Let $v(x) = \ln x$, then

since $v = \ln x$ this $= \frac{1}{x}$

$$y' = \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dv}{dv} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{dy}{dv} \cdot \frac{1}{x}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dv}\right)$$

product rule

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dv}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x} \cdot \frac{d}{dx} \cdot \frac{dv}{dv} \frac{dy}{dv}$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d^2y}{dv^2} \cdot \frac{dv}{dx}$$

$\frac{1}{x}$ since $v = \ln x$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}.$$

$$\Rightarrow y'' = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}$$

Plug y, y', y'' into Eq (3), we have

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}\right) + 2x \cdot \frac{1}{x} \cdot \frac{dy}{dv} - 12y = 0$$

$$\Rightarrow -\frac{dy}{dv} + \frac{d^2y}{dv^2} + 2\cdot\frac{dy}{dv} - 12y = 0$$

$$\Rightarrow \frac{d^2y}{dv^2} + \frac{dy}{dv} - 12y = 0$$

This is in the form $ay'' + by' + cy = 0$, where y is a function of v.

The char. eqn is

$$r^2 + r - 12r = 0$$

$$\Rightarrow (r+4)(r-3) = 0$$

$$\Rightarrow r_1 = -4,\quad r_2 = 3 \text{ (distinct real roots).}$$

So $y = c_1y_1 + c_2y_2 = c_1e^{-4v} + c_2e^{3v}$

backsubs $v = \ln x$

$$= c_1e^{-4\ln x} + c_2e^{3\ln x}$$

$$\Rightarrow y(x) = c_1x^{-4} + c_2x^3$$

This is the general solution to Eq(3).

Example 4. Recall a **second-order Euler equation** is one of the form

$$ax^2y'' + bxy' + cy = 0 \tag{2}$$

where a, b, c are constants.

(a) Show that if $x > 0$, then the substitution $v = \ln x$ transforms Eq. (2) into the constant coefficient linear equation

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0 \tag{3}$$

with independent variable v.

(b) If the roots r_1 and r_2 of the characteristic equation of Eq. (3) are **real and distinct**, conclude that a general solution of the Euler equation in (3) is $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

ANS: (a) Let $v = \ln x$, then

since $v = \ln x$ this $= \frac{1}{x}$

$$y' = \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dv}{dv} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{dy}{dv} \cdot \frac{1}{x}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dv}\right)$$

product rule

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dv}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\cdot\frac{d}{dx}\cdot\frac{dv}{dv} \rightarrow \frac{dy}{dv}$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d^2y}{dv^2}\cdot\frac{dv}{dx}$$ $\frac{1}{x}$ since $v = \ln x$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}.$$

$$\Rightarrow y'' = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}$$

Plug y, y', y'' into Eq (2), we have (leave it as exercise)

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0.$$

(b) If r_1 r_2 are real and distinct.

$$y(x) = c_1 e^{r_1 v} + c_2 e^{r_2 v} = c_1 (e^v)^{r_1} + c_1 (e^v)^{r_2}$$

$$= c_1 (e^{\ln x})^{r_1} + c_2 (e^{\ln x})^{r_2}$$

$$= c_1 x^{r_1} + c_2 x^{r_2}$$

Exercise 5. Find y as a function of x if

$$x^2y'' + 4xy' - 10y = 0,$$

$$y(1) = 4, \quad y'(1) = -7$$

ANS: Method 1. Directly apply the formula in Example 4.

Compare the given eqn with Eq (2) in Example 4.

We know $a=1$, $b=4$, $c=-10$.

Thus if we assume $v=\ln x$, then

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0$$ (with $a=1$, $b=4$, $c=-10$)

$$\Rightarrow \frac{d^2y}{dv^2} + 3\frac{dy}{dv} - 10y = 0.$$

This is of the form $y''+3y'-10y=0$ where y is a function in terms of v.

The char. eqn is

$$r^2+3r-10=0$$

$$\Rightarrow (r-5)(r+2)=0$$

$$\Rightarrow r_1=5, \quad r_2=-2.$$

Thus $y(x) = c_1y_1 + c_2y_2 = c_1e^{-5v} + c_2e^{2v} = c_1e^{-5\ln x} + c_2e^{2\ln x}$

$= c_1x^{-5} + c_2x^2$

As $y(1)=4$, $4=c_1+c_2$ ①

Compute $y'=-5c_1x^{-6}+2c_2x$

$y'(1)=-7$ implies $-7=-5c_1+2c_2$ ②

Combining ① ②, we have

$$\begin{cases} c_1+c_2=4 \\ -5c_1+2c_2=-7 \end{cases} \Rightarrow \begin{cases} c_1=\frac{15}{7} \\ c_2=\frac{13}{7} \end{cases}$$

Thus

$$y(x)=\frac{15}{7}x^{-5}+\frac{13}{7}x^2$$

Method 2. Assume $v=\ln x$ and compute y' y'' in terms of $v=\ln x$ (without knowing the formula in Example 4)

We have (I'll skip the steps as the first part is identical to Example 3 and Example 4)

$$y'=\frac{1}{x}\frac{dy}{dv}$$

$$y''=-\frac{1}{x^2}\frac{dy}{dv}+\frac{1}{x^2}\frac{d^2y}{dv^2}$$

Plug them into the given eqn (4), we have

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dv}+\frac{1}{x^2}\frac{d^2y}{dv^2}\right)+4x\left(\frac{dy}{dv}\cdot\frac{1}{x}\right)-10y=0$$

$$\Rightarrow -\frac{dy}{dv}+\frac{d^2y}{dv^2}+4\frac{dy}{dv}-10y=0$$

$$\Rightarrow \frac{d^2y}{dv^2}+3\frac{dy}{dv}-10y=0$$

The rest part is the same as Method 1.